Wavefront Evolution in Various 2-dimensional Geometries

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Consider a smooth curve γ in the Euclidean plane. We can consider it as an initial wavefront, subsequent wavefronts are equivalently the envelopes of the circles of fixed radius centred on γ, or tangent to γ, or the points traced out along the normals a fixed distance from the corresponding point on γ. The singular points of these wavefronts trace out the caustic, which can be thought of as the centres of the osculating circles to γ or the envelope of the normals to γ. This caustic has singularities at centres of curvature corresponding to vertices (points where the curvature has an extremum). The question of how the wavefronts generically evolve along the caustic and through these singular points was solved by Arnold in 1976. They evolve as generic sections of the swallowtail surface; indeed he solves the problem of wavefront evolution in higher dimensional spaces too. However there is an unsatisfactory aspect to the solution. We can describe how the wavefronts evolve, but not how they are distributed with respect to the caustic; indeed there is no smooth model. Nevertheless intuitively there are clearly two possible types of generic transition, this paper we make precise sense of this using a new type of universal picture linked to versal unfoldings of functions.

Arnold's work describes the generic transitions of Legendrian singularities; wavefronts in Euclidean space are one example, but they have their own special characteristics, for example the cuspidal tangent to the wavefront at a point of the evolute is orthogonal to the evolute. This means, for example, that some of the transitions in Arnold's list do not occur in dimension 3. It also means that in dimension 2 the transition through a cusp on the caustic is always generic; there is no additional condition to be satisfied. It is an observation of Farid Tari that the transitions observed are always just one of the types; we give another proof of this here. One can also consider wavefront transition in other types of 2-dimensional geometries; in particular in Minkowski plane with an indefinite metric. The corresponding geometry has been worked out by Tari and his colleagues. It turns out that wavefront transition here is always of the other type. We also apply these results to implicit differential equations.